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OPTIMAL CONTROL OF PARABOLIC PDE BY A COMBINED ADOMIAN/ALIENOR MODEL

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Abstract. This article presents a numerical solution to the optimal control problem governed by linear parabolic differential equation (PDE), generally solved by direct numerical method. It consists in choosing the controls in a finite dimension space (piecewise constants) by the discretization of the time interval. The optimal control problem is approached by an optimisation problem under constraints. To solve this problem, two combined mathematical methods are used : the Adomian decomposition method and the Alienor method. Firstly, the Adomian method is used to solve the PDE by explicitly expressing the solution as convergence series; depending on controls and requires no discretization in space and in time, contrary to numerical methods. Secondly, the Alienor method reduces the problem of minimization of a function with several variables to a function with one variable, unlike the iterative optimisation methods which require initialization of the initial vector. An application of this Adomian /Alienor combined model to solve the PDE is performed, and results are compared with those obtained by Adomian/Levenberg-Marquardt method.

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Keywords and phrases. Optimal control, Parabolic PDE, Adomian/Alienor model, A direct numerical method of optimal control.

1. Introduction

Most of the phenomena in real world (physics, mechanics, biology, economics and finance) are described through mathematical models in a form of Partial Differential Equations (PDEs).

It is often a matter of studying the possibility of acting on the system in order to determine the way it operates best to attain a desired goal, otherwise stabilising is to make it insensitive under some disturbances. This is the subject of control theory. In biology, the optimal control can be applied to concrete real case studies such as controlling the growth of some populations (cancer cells, bacteria, viruses... etc.) using chemical treatments, such as cancer chemotherapy. So, the objective here is to minimise the number of cancer cells with an optimal therapeutic doses.

The problem of optimal control systems governed by partial differential equations (PDE) has been extensively studied in the literature (see [1],[2],[3]). J.Lions [1] has given the methods to solve the optimal control problem of systems governed by PDE.

In [4], two numerical methods for solving the optimal control problem are presented : the direct methods, and indirect methods. The direct methods consist of discretising the state variable and control, then approximating the optimal control problem to an optimisation problem. The indirect methods involve solving the problem numerically using a shooting method, and the problem of boundary values is obtained by applying the maximum principle.

The presented work proposes to use a direct method to compute the optimal control of a system governed by a linear parabolic PDE. The controls are chosen in a finite dimensional space (piecewise constant functions), the Adomian decomposition method (ADM) is applied to approximate the solutions of the proposed equation and combined with the Alienor method to resolve the optimisation problem. In ([5],[6],[11]), the ADM is used to solve the linear and nonlinear systems (differential, partial, algebraic, integral,...etc.), where the solution is an analytical function given as explicit series forms dependent on the parameters. This method is based on the decomposition of the nonlinear part of the system, using special polynomials called Adomian polynomials which are calculated by recursive formulas [5]. The solution of the parabolic PDE is given as small intervals of time, and the optimal control is reduced to a minimised problem, of "n" variables. We have used the Alienor method ([5],[14],[9]) to reduce the optimisation problem of n variables at one variable optimisation problem. This method is based on the reduced transformation to constuct the densities curves of $\mathbb{R}^n$ space. Therefore, the combination of these two methods can transform the optimal control problem of a system governed by PDE into a minimisation problem of a function at single variable.

This article is organised as follows, in the second section we present the formulation of optimal problem governed by a parabolic PDE. In section 3, we developed, the mathematicals methods of ADM and Alienor. Section 4 presents a numerical method for solving the optimal control problem governed by a parabolic equation, and demonstrates how the optimal control problem of parabolic PDE is reduced to a classical constrained optimisation problem of one variable problem by using the combined method Adomian/Alienor. An application of problem to the equation is given in section 5, and we conclude with a comparative study of the results.

2. Problem statement

Consider the linear parabolic equation with a control function $q(t)$ [12] :

$$\frac{\partial V}{\partial t} = \alpha \frac{\partial^2 V}{\partial x^2} + qV \quad \text{in } (x,t) \in \Omega \text{ x }]0,T[\tag{1}$$

$$V(x,0) = f(x), \quad x \in \Omega \tag{2}$$

$$V(x,t) = 0 \qquad , x \in \partial\Omega, \; 0 < t < T \tag{3}$$

Where :
Ω : is a bounded domain of $\mathbb{R}^n$ (n=1,2,3),
$V(x,t)$ is a state function,
α is a non-null coefficient,
f is a strictly positive function on Ω,
$q(t)$ is a function of control.
The problem (1)-(2)-(3) has a positive non-null solution [10].

Often, these equations appear in different real phenomena, like a substance diffusing in chemistry, water pollution (river, layerphrenic,...) studies. Also in biology, the dynamics of gas in the breathing process of man and animal [5]...etc.

In biology, for example, the control $q(t)$ represents the amount of the therapeutic drug to affect the tumour during the interval [0,T], where T is set a priori. In this phenomenon, the optimal control problem can be reformulated as follows:

We seek out the control $q(t)$ solution of :

$$\underset{q\in Q}{Min} J(q) \tag{4}$$

where :

$$J(q) = \int_{w}\int_{0}^{T} g(V(x,t),q(t))dxds$$

and g and T are known, w is a subdomain of Ω.

The set of feasible controls Q may be a space or a closed convex sub set.

$V(x,t,q)$ denotes the solution with a parameter q of the system (1)-(2)-(3).

It can be assumed that $q(t)$ is bounded and satisfies the inequation (5):

$$a \leqslant q(t) \leqslant b \tag{5}$$

where $a, b \in \mathbb{R}^{+}$.

Often, this problem is solved by traditional numerical methods : direct methods or indirect methods [4]. We have proposed,in our work, a direct method using the combined mathematical methods of Adomian decomposition and Alienor.

3. Mathematicals methods

3.1. **Adomian decomposition method.** This method is used to solve linear and nonlinear functional equations of different kinds : differential, boundary value problem, integrals, algebraic,... etc.

Consider the following functional equation [5]:

$$x - \mathbf{N}(x) = g \tag{6}$$

where $\mathbf{N}$ represents a non-linear operator (differential, boundary value problem, integral, ...), g is a known function, and x is the solution of (6).

The ADM develops the solution x (if it exists) into the series form below :

$$x = \sum_{i=0}^{\infty} x_i \tag{7}$$

moreover, the non-linear operator $\mathbf{N}(x)$ can be decomposed into the following series form :

$$\mathbf{N}(x) = \sum_{i=0}^{\infty} A_i(x_0, x_1, ..., x_i) \tag{8}$$

where A_i are the Adomian polynomials dependent on $x_0, x_1, ..., x_i$ (see [5]). It is assumed that these two series are convergent.

By putting the expressions (7) and (8) into (6), we get :

$$\sum_{i=0}^{\infty} x_i - \sum_{i=0}^{\infty} A_i = g \tag{9}$$

By identifying the two sides of equation (9), it yields :

$$\begin{aligned} x_0 &= g \\ x_1 &= A_0(x_0) \\ x_2 &= A_1(x_0, x_1) \\ &\vdots \\ x_{i+1} &= A_i(x_0, ..., x_i) \end{aligned} \tag{10}$$

We can easily calculate the series terms of the series x_i of our solution, we have just gives the polynomials Adomian, by the following expressions [5]:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n} N(\sum_{i=0}^{n} \lambda^i x_i)|_{\lambda=0}, n = 0, 1, 2, ...$$

Y.Cherruault and K.Abbaoui have proved that the series $\sum x_i$ converges if the non-linear operator **N** satisfies certain conditions (see [5]). Adomian polynomials exist and the series $\sum A_i$ converges ([5]), practical formulas for these polynomials are proposed (see [5], [8]).

3.2. **Alienor method.** The multidimensional Alienor global optimisation method has been elaborated in the 1980s by Cherruault and Guillez ([7],[8]). The Alienor method is based on the idea of reducing a several variables minimisation problem to a single variable minimisation problem allowing the use of well-known powerful methods and techniques available in the case of a single variable.

The basic idea of this method is the densification of space $\mathbb{R}^n$ by so-called α-denses curves.

Let us first recall a definition of α- density.

Definition:

A curve h defines : $A = [0, M] \to \prod_{i=1}^{n} [a_i, b_i]$

is called $\alpha - dense$ in $\prod_{i=1}^{n} [a_i, b_i]$, if for any $w \in \prod_{i=1}^{n} [a_i, b_i]$ there exists $\theta \in A$ such that :

$$d(w, h(\theta)) \leqslant \alpha \tag{11}$$

where d is the Euclidean distance in $\mathbb{R}^n$. The number α is a real strictly positive and assumed very small compared to the dimensions of the hyper-rectangle $\prod_{i=1}^{n} [a_i, b_i]$.

The Alienor reducing transformation method can be summarized as follows. It is asked for solving the global minimisation problem :

$$\underset{x_1, x_2, \ldots, x_n}{Min} \; J(x_1, x_2, \ldots, x_n) \tag{12}$$

where J is continuos function on $\mathbb{R}^n$.

We seek out the global minimum of J, such that it satisfies the following condition :

(13) $$\lim_{x_1^2+...+x_n^2 \to \infty} J(x_1, \ldots, x_n) = +\infty$$

By putting the variables x_i, $i = 1, .., n$ expresses as follows, into the problem (12):

(14) $$x_i = h_i(\theta) \quad \theta \geq 0, \, i = 1, \ldots, n$$

where $h_i(\theta) \in C^\infty$ are the functions called reducing transformation, such that a parameterized curve $h(\theta) = (h_1(\theta), ..., h_n(\theta))$ is α-dense in $\prod_{i=1}^{n} [a_i, b_i]$. Consequently, the minimisation problem (12) is then approximated by the problem of minimisation at single variable :

(15) $$\underset{\theta \in [0, \theta_{\max}]}{Min} J^*(\theta)$$

where $J^*(\theta) = J(h_1(\theta), h_2(\theta), \ldots, h_n(\theta))$. θ_{max} is the supremum of the definition domain of the function h when it α-densities the hyperrectangle.

In the basic method, the unidimensional minimisation problem (15) is solved by discretizing the interval $[0, \theta_{\max}]$ via a chosen step $\Delta\theta$. Then we look for the minimum of the finite set $\{J^*(\theta_k), k = 0, 1, ..., N\}$ where θ_0, θ_1,..., θ_N are the discretized points. Obviously, the densification parameter α and the step $\Delta\theta$ are chosen in such a way that the global minimum is obtained with the desired accuracy ε [15].

Y.Cherruault in 1999 [8], has proved that any solution of (15) is an approximation of the solution of (12). We can ensure that the global minimum of $J^*(\theta)$ is a good minimum approximation of problem (12) via the local variations method [5].

It is possible to have the global minimum of a functional at single variable by the Optimisation Preserving Operators (O.P.O) [9] in order to avoid finding the local minima and to interest only on the global minimum.

4. Numerical method of optimal control

A numerical direct method ([4]) is presented in two stages to solve the optimal control problem governed by a linear parabolic PDE. The first step consists of transforming the optimal control into a constrained optimisation problem. In the second stage, we solve this problem by an appropriate method.

The aim of this presented work is to combine two methods : ADM and Alienor. They allow to reduce the multidimensional minimisation problem depend explicitely on the controls to unidimensional minimisation problem.

The procedure is as follows :

Recall that it is possible to determine the optimal control $q(t)$ minimising the following creteria :

(16) $$\underset{q \in Q}{Min} J(q)$$

where : $J(q) = \int_w \int_0^T g(V(x,t), q(t)) dxds$

$V(t)$ and $q(t)$ satisfy the following PDE :

(17) $$\begin{cases} \frac{\partial V(x,t)}{\partial t} = \alpha \frac{\partial^2 V(x,t)}{\partial x^2} + qV(x,t) \text{ dans } \Omega \text{ x }]0,T[\\ V(x,0) = f(x) \quad , x \in \Omega \end{cases}$$

We subdivide the interval $[0,T]$ on time subintervals of uniform size Δt . Given a parameter N $(N > 0)$,we set:

$$\Delta t = \frac{T}{N}$$

and denotes by $t_k = k.\Delta t$ for $k = 0, ..., N$.

The control $q(t)$ is approximated by piecewise constants values on each interval $[t_k, t_{k+1}]$:

(18) $$q(t) = q_k, \quad t \in [t_k, t_{k+1}] \text{ and } \quad k = 0, ..., N-1$$

$q(t)$ is a feasible control satisfying the following constraint :

(19) $$a \leqslant q(t) \leqslant b$$

For q_k verifying (19), we get :

(20) $$a \leqslant q_k \leqslant b, \quad \text{for } k = 0, ..., N-1$$

We use the ADM on each sub-interval $[t_k, t_{k+1}]$ to solve the equation (17). By substituting $q(t)$ by the formulae (18) into (17), then integrating between t_k and t_{k+1}, we obtain the terms of solution [16]:

(21)
$$V_n(x,t,q_0,..,q_k) = \sum_{p=0}^{n} C_n^p \alpha^{n-p} q_k^p (L_t^{-1})^n L_{xx}^{(n-p)} V(x,t_k) = (L_t^{-1})^n (\alpha L_{xx} + q)^n V(x,t_k)$$

where $V(x,t_0) = f(x)$, $V(x,t_k)$ depends of controls $q_0, q_1, .., q_{k-1}$ and $(L_t^{-1})^n = \int_0^t ... \int_0^r \int_0^\tau (.)\, ds\, d\tau\, dr$ is the n th integration order.

The truncated Adomian series in interval $[t_k, t_{k+1}]$ is given as follows :

(22)
$$V^{(k)}(x,t,q) = \sum_{n=0}^{s} V_n(x,t,q_0,..,q_k) = \sum_{n=0}^{s} \frac{(t-t_k)^n}{n!} (\alpha L_{xx} + q_k)^n V(x,t_k,q_0,q_1,..,q_k)$$

This solution explicitly depends on $q_0, q_1, .., q_k$.

The requirements reattachment of the series solution are needed to proceed at the initialized of the next step, then the first term of the Adomian solution is calculated as follows :

(23) $$V_0(x, t = t_k) = V^{(k)}(x, t = t_k)$$

By substituting the expression (22) into the objective function (16), we get the following approximation :

(24) $$J \approx \sum_{k=0}^{N-1} g_k(q_0, q_1, \cdots, q_k)$$

where

$$g_k(q_0, q_1, \cdots, q_k) = \int_w \int_{t_k}^{t_{k+1}} g(V^k(x, s, q_0, q_1, \cdots, q_k)) dx ds$$

with $N.\Delta t = T$.
The optimal control is therefore approximated by :

$$\underset{q_0,q_1,\ldots,q_{N-1}}{Min} \quad J \approx \underset{q_0,q_1,\ldots,q_{N-1}}{Min} \quad \sum_{k=0}^{N-1} g_k(q_0, q_1, \ldots, q_{N-1}) \tag{25}$$

$$Such\ as\ \ q_k \in [a, b] \tag{26}$$

It is a minimisation problem with N unknown variables on $[a, b]$. These variables can be reduced by Alienor (or variants) with a single variable θ. The Alienor transformations are defined for parameters of control as follows:

$$q_k = h_k(\theta),\ \ k = 0, ..., N-1 \tag{27}$$

The transformation $h_j(\theta)$ is chosen such a way that it densifies the space $\mathbb{R}^N$.

Substituting (27) into the function (25), the global minimum problem of the problem (25) is approached by a minimisation problem of a function with one variable :

$$\underset{\theta}{Min} J^*(\theta) \tag{28}$$

where : $J^*(\theta) = \sum\limits_{k=0}^{N-1} g_k(h_0(\theta), ..., h_{N-1}(\theta))$

The function $J^*(\theta)$ is continuous on $[\ 0,\ \theta_{\max}]$ and attain at least one minimum on $[\ 0,\ \theta_{\max}]$.

5. Application to the parabolic equation

5.1. **Problem statment.** Consider the following linear parabolic PDE [12]:

$$\frac{\partial V(x,t)}{\partial t} = \alpha \frac{\partial^2 V(x,t)}{\partial x^2} + qV(x,t)\ ,\ (x,t) \in]0,1[\ \text{x}\ ]0,T[\tag{29}$$

$$V(x,0) = f(x), \quad x \in]0,1[\tag{30}$$

$$V(0,t) = 0 \qquad ,0 < t < T \tag{31}$$

where $V(x,t)$ is the concentration of a chemical substance may diffuse into the blood through the lung-blood interface (or kidny, liver,..etc). The unkown control variable $q(t)$ of the equation (29) is a therapeutic drug. Assume that the state equation has a positive single solution [10].

We seek the optimal drug therapeutic $q(t)$ minimising the objective function :

$$J = \int_0^T (V(x^*,t) - d)^2 dt \tag{32}$$

where $0 \leqslant q(t) \leqslant 1$, d is a given positive constant and f is a given positive function and continuous on $[0,1]$.
For a given "d"; the method consists of finding a control function $q(t)$ so that the concentration $V(x,t)$ of a chemical substance remains close to the value"d" during $[0,T]$ at the point x^*.

In therapy, $q(t) = 1$ designates the maximum dose which can create undesirable side effects on the human organism, however the low doses ($q(t) \approx 0$) will have no effect on the disease. We seek the optimal therapy, in order to maintain an acceptable level the concentration of a chemical substance in the blood taking into account side effects experienced by the patient.

5.2. **Numerical results and discussion.** We seek out to determine the control function $q(t)$ minimising the criterion (32). The constants are fixed : $d = 1$, $\alpha = 0.015$, $T = 1$, $x^* = 0.5$. The initial condition is given by : $f(x) = x$, $0 < x < 1$.

We divide the time interval $[0, 1]$ into $N = 5$ subintervals of equal length, $\Delta t = 0.2$. The control function $q(t)$ is approximated in the PDE (29) by the constants q_k satisfying :

$$0 \leqslant q_k \leqslant 1, \quad k = 0, .., 4.$$

This PDE is solved by the ADM on the interval $[0, T]$.The truncated Adomian solution at the odrer 2, depends explicitly on $q_0, q_1, ..., q_4$ and the time variable t. This solution is substituted into the function (32), consequentely:

$$\underset{q_0,q_1,\ldots,q_4\in[0,1]}{Min} J \approx \underset{q_0,q_1,\ldots,q_4\in[0,1]}{Min} \sum_{k=0}^{N-1} g_k(q_0, q_1, \ldots, q_4) \tag{33}$$

The problem (33) is minimised by the Alienor method with the following transformation :

$$\begin{aligned} q_0 &= \theta \\ q_k &= \frac{1}{2}(1 - \sin(2^k \pi\theta)) \ , \quad k = 1, .., 4 \end{aligned}$$

- **The numerical results**

A similar study to this work was realised by Messaoudi and Manseur ([13]) to determine the optimal control of linear parabolic PDE, by minimising 5 times an objective function at a single variable "q_k", over each time interval $[t_k, t_{k+1}]_{k=0,\ldots,4}$.

The table 1 presents a comparaison of results obtained by the combined Adomian/Alienor method, Adomian/ $\underset{q_k\in[\,0,\,1]}{Min} J(q_k)$ (see [13]) with those of Adomian/Levenberg-Marquardt for initial values of solution [0.8,0.5,0.8,0.7,0.01]:

Table 1. Values constants of control and the objective function

q_0	q_1	q_2	q_3	q_4	Valeur de J	Methods
0.9	0.795	0.976	0.789	0.028	0.0782	Adomian/Alienor
1	1	1	0.714	0	0.069	Adomian/ $\underset{q_k\in[\,0,\,1]}{Min} J(q_k)$ [13]
1	1	1	0.65	0	0.068	Adomian/ Levenberg-Marquardt

The approximated values of optimal control are substitute in the solutions of the linear parabolic PDE given by ADM to compare them with the solutions of the equation when the control is maximum $q(t) = 1$.

The superposition curves of PDE solutions $V(x, t, q)$ obtained by Adomian/Alienor method (Black color), Adomian /Levenberg-Marquardt (in

Red) and Adomian/ $\underset{q_k \in [\,0,\,1]}{Min} J\,(q_k)$ [13](in green) with the maximum dose ($q = 1$, blue points) at position x^* is shown in Figure 1. From this figure, we note that the optimal state $V(x, t, q_{opt})$ at the position x^* is maintained around the desired value "$d = 1$".

The curve of the optimum control values obtained by the three methods for a time step equal to $\Delta t = 0.2$ is shown in Figure 2.

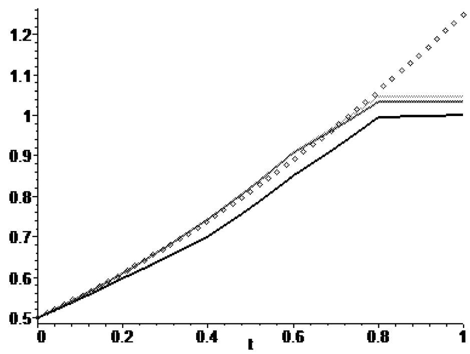

FIGURE 1. Superposition of solutions curves $V(x, t, q^*)$ at point $x^* = 0.5$

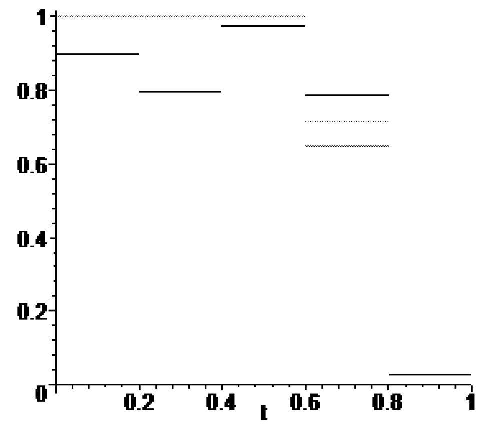

FIGURE 2. Function of optimal controls

Discussion

We have used two optimisation methods to find the optimal control : Alienor method and Levenberg-Marquardt method.

From the results shown in Table 1, it is found that the value of J given by Levenberg-Marquardt after 9 iterations is minimum and the control values are close to those obtained by Alienor.

The Levenberg-Marquardt method depends on the choice of initial values of the control. When this choice is arbitrary, this method does not yield good results. If the starting values are well chosen, it makes a small number of iterations, since at a large number of iterations we risk to diverge from the optimal solution. All these drawbacks do not exist in the Alienor method.

Our method is more efficient than the one given by approach of [13], because we seek out the global minimum on the all interval $[0, T]$, however the approach of [13] gives a global minimum on each sub interval of time $[t_k, t_{k+1}]$.

Both approaches have achieved the desired objective. Now it is up to the therapist to choose patient treatment. If he wanted to give treatment on a period prescribed by the doctor, he would choose our methodology. Or, if we calculate the optimum dose before each take of treatment on a small interval of time , knowing this previous dose we can improve the next dose and so on , up to the completion of treatment period, which corresponds to the approach [13].

6. Conclusion

The optimal control problem of linear parabolic PDEs was investigated, using a methodology based on the combination of Adomian and Alienor methods. The method Adomian can express the solution of the equation in the form of convergent series explicitly dependent on the controls and that it requires no discretization in space and time relative to the finite difference and elements finite numericals methods. Alienor method reduces the problem of minimisation of several variables function with a problem of minimisation of a single variable function. The combination of these two methods can reduce the linear parabolic PDE control problem to an optimisation problem of a single variable function.

An application to the problem of optimal therapy governed by parabolic PDE is carry out and compared with other methods and the results obtained are satisfying. According to the physical condition of a patient, we can determine the corresponding optimal treatment.

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